Dromion solutions of noncommutative Davey–Stewartson equations

2009 J. Phys. A: Math. Theor. 42 235202

(http://iopscience.iop.org/1751-8121/42/23/235202)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 235202 (20pp) doi:10.1088/1751-8113/42/23/235202

Dromion solutions of noncommutative Davey–Stewartson equations

C R Gilson and S R Macfarlane

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK

E-mail: s.macfarlane@maths.gla.ac.uk

Received 17 February 2009, in final form 24 April 2009
Published 15 May 2009
Online at stacks.iop.org/JPhysA/42/235202

Abstract
We consider a noncommutative version of the Davey–Stewartson equations and derive two families of quasideterminant solution via Darboux and binary Darboux transformations. These solutions can be verified by direct substitution. We then calculate the dromion solutions of the equations and obtain computer plots in a noncommutative setting.

PACS numbers: 02.30.Ik, 02.40.Gh

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Davey–Stewartson (DS) equations have become a topic of much interest in recent years. Derived by Davey and Stewartson in 1974 [1], the system is nonlinear in $(2 + 1)$-dimensions and describes the evolution of a three-dimensional wave-packet on water of finite depth. By carrying out a suitable dimensional reduction, the system can be reduced to the $(1 + 1)$-dimensional nonlinear Schrödinger (NLS) equation.

A major development in the understanding of the DS equations came in 1988, when Boiti *et al* [2] discovered a class of exponentially localized solutions (two-dimensional solitons) which undergo a phase shift and possible amplitude change on interaction with other solitons. These were later termed *dromions* by Fokas and Santini [3], derived from the Greek *dromos* meaning *tracks*, to highlight that the dromions lie at the intersection of perpendicular track-like plane waves.

Multidromion solutions to the DS system have been obtained using a variety of approaches—the inverse scattering method [3], Hirota's direct method [4] and others. These solutions have been determined both in terms of Wronskian [4] and Grammian [5] determinants.

Additionally, there has been considerable interest in noncommutative versions of the DS equations. Hamanaka [6] derived a system with noncommutativity defined in terms of the

Moyal star product [7], while more recently, Dimakis and Müller-Hoissen [8] determined a similar system from a multicomponent KP hierarchy. This then enabled calculation of dromion solutions in the matrix case.

The strategy that we employ here, whereby we introduce noncommutativity into an integrable nonlinear wave equation without destroying the solvability, has previously been considered by others in the field, for example by Lechtenfeld and Popov [9], and by Lechtenfeld, Popov *et al* in [10], where a noncommutative version of the sine-Gordon equation is discussed.

In this paper we are not concerned with the nature of the noncommutativity, and derive a system of noncommutative DS equations in the most general way by utilizing the same Lax pair as in the commutative case but assuming no commutativity of the dependent variables. This method has also been employed by Gilson and Nimmo in [11] for the case of the noncommutative Kadomtsev–Petviashvili (KP) equation. We find that the noncommutative DS system obtained in this manner corresponds to that given in an earlier paper by Schultz, Ablowitz and Bar Yaacov [12], where a quantum version of the DS equation is ultimately discussed.

We derive quasi-Wronskian and quasi-Grammian solutions of this system in section 4 via Darboux and binary Darboux transformations and, in section 6, verify these solutions by direct substitution.

We then use the quasi-Grammian solution to determine a class of dromion solution and, by specifying that certain parameters in the solution are of matrix rather than scalar form, obtain dromion solutions in the noncommutative case. We conclude with computer plots of these dromion solutions.

2. Noncommutative Davey–Stewartson equations

We consider the system of commutative DS equations given by Ablowitz and Schultz in [13], with Lax pair

$$L = \partial_x - \Lambda + \sigma J \partial_y, \tag{2.1a}$$

$$M = \partial_t - A + \frac{\mathrm{i}}{\sigma}\Lambda\partial_y - \mathrm{i}J\partial_{yy}, \tag{2.1b}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0 & q(x,y,t) \\ r(x,y,t) & 0 \end{pmatrix} \tag{2.2}$$

for $r = \pm q^*$ (q^* denotes the complex conjugate of q) and A is a 2×2 matrix given by

$$A = \begin{pmatrix} A_1 & \dfrac{\mathrm{i}}{2\sigma^2}(q_x - \sigma q_y) \\ -\dfrac{\mathrm{i}}{2\sigma^2}(r_x + \sigma r_y) & A_2 \end{pmatrix}. \tag{2.3}$$

We choose $\sigma = -1$ or $\sigma = \mathrm{i}$ for the DSI and DSII equations, respectively. By considering the same Lax pair (2.1a), (2.1b) as is used in the commutative case and assuming no commutativity of variables (we do not specify the nature of the noncommutativity), we obtain the compatibility condition

$$-A_x + [\Lambda, A] + \Lambda_t - \sigma J A_y + \frac{\mathrm{i}}{\sigma}\Lambda\Lambda_y - \mathrm{i}J\Lambda_{yy} = 0, \tag{2.4a}$$

$$\frac{\mathrm{i}}{\sigma}\Lambda_x + \sigma[A, J] - \mathrm{i}J\Lambda_y = 0, \tag{2.4b}$$

from which we generate a system of noncommutative Davey–Stewartson (ncDS) equations

$$\mathrm{i}q_t = -\frac{1}{2\sigma^2}(q_{xx} + \sigma^2 q_{yy}) + \mathrm{i}(A_1 q - qA_2), \tag{2.5a}$$

$$\mathrm{i}r_t = \frac{1}{2\sigma^2}(r_{xx} + \sigma^2 r_{yy}) - \mathrm{i}(rA_1 - A_2 r), \tag{2.5b}$$

$$(\partial_x + \sigma\partial_y)A_1 = -\frac{\mathrm{i}}{2\sigma^2}(\partial_x - \sigma\partial_y)(qr), \tag{2.5c}$$

$$(\partial_x - \sigma\partial_y)A_2 = \frac{\mathrm{i}}{2\sigma^2}(\partial_x + \sigma\partial_y)(rq). \tag{2.5d}$$

(Note that we obtain from the above system the nonlinear Schrödinger (NLS) equation [14]

$$\mathrm{i}q_t + q_{yy} \pm 2qrq = 0 \tag{2.6}$$

and its corresponding complex conjugate by taking a dimensional reduction $\partial_x = 0$, with $A_1 = \pm\frac{\mathrm{i}}{2}qr$, $A_2 = \pm\frac{\mathrm{i}}{2}rq$ [6]).

For notational convenience, and to avoid the use of identities later when verifying solutions, we introduce a 2×2 matrix $S = (s_{ij})$ $(i, j = 1, 2)$ such that $\Lambda = [J, \sigma S]$ [15], and hence

$$S = \begin{pmatrix} s_{11} & \dfrac{q}{2\sigma} \\ -\dfrac{r}{2\sigma} & s_{22} \end{pmatrix}. \tag{2.7}$$

Additionally, by setting

$$A = \frac{\mathrm{i}}{\sigma}S_x - \mathrm{i}JS_y, \tag{2.8}$$

(2.4*b*) is automatically satisfied, and (2.4*a*) becomes

$$-\frac{\mathrm{i}}{\sigma}S_{xx} + \mathrm{i}JSS_x - \mathrm{i}SJS_x - \mathrm{i}S_xJS + \mathrm{i}S_xSJ + \mathrm{i}\sigma JS_yJS \\ - \mathrm{i}\sigma JS_ySJ + \sigma JS_t - \sigma S_tJ - \mathrm{i}\sigma JSS_yJ + \mathrm{i}\sigma SJS_yJ + \mathrm{i}\sigma JS_{yy}J = 0. \tag{2.9}$$

Note that this is essentially the noncommutative analogue of the Hirota bilinear form (see, for example, [16]) of the DS equations.

3. Quasideterminants

Here we briefly recall some of the properties of quasideterminants. A more detailed analysis can be found in the original papers [17, 18].

The notion of a *quasideterminant* was first introduced by Gelfand and Retakh in [18] as a straightforward way to define the determinant of a matrix with noncommutative entries. Many equivalent definitions of quasideterminants exist, one such being a recursive definition involving inverse minors. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries over a usually noncommutative ring $\mathcal{R}$. We denote the (i, j)th quasideterminant by $|A|_{ij}$, where

$$|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} s_j^i. \tag{3.1}$$

Here, A^{ij} is the $(n-1)\times(n-1)$ minor matrix obtained from A by deleting the ith row and the jth column (note that this matrix must be invertible), r_i^j is the row vector obtained from the ith row of A by deleting the jth entry, and s_j^i is the column vector obtained from the jth column of A by deleting the ith entry.

A common notation employed when discussing quasideterminants is to 'box' the expansion element, i.e. we write

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{3.2}$$

to denote the (1, 1)th quasideterminant. It should be noted that the above expansion formula is also valid in the case of block matrices, provided the matrix to be inverted is square; for example considering a block matrix

$$\begin{pmatrix} N & B \\ C & d \end{pmatrix},$$

where N is a square matrix over $\mathcal{R}$, B and C are column and row vectors over $\mathcal{R}$ of compatible lengths, and $d \in \mathcal{R}$, we have

$$\begin{pmatrix} N & B \\ C & \boxed{d} \end{pmatrix} = d - CN^{-1}B. \tag{3.3}$$

Quasideterminants also provide a useful formula for the inverse of a matrix: for an invertible $n \times n$ matrix $A = (a_{ij})$ $(i, j = 1, \ldots, n)$, the (i, j)th entry of A^{-1} is given by

$$(A^{-1})_{ij} = (|A|_{ji})^{-1}. \tag{3.4}$$

When the elements of A commute, the quasideterminant $|A|_{ij}$ is not simply the determinant of A, but rather a *ratio* of determinants: it is well known that, for A invertible, the (j, i)th entry of A^{-1} is

$$(-1)^{i+j} \frac{\det A^{ij}}{\det A}.$$

Then, by (3.4), we can easily see that

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}} \tag{3.5}$$

in the commutative case.

4. Quasideterminant solutions via Darboux transformations

4.1. Darboux transformations

Here we give a brief overview of Darboux transformations. Further information can be found in, for example, [19].

We follow the notation given in [11]. Let $\theta_1, \ldots, \theta_n$ be a particular set of eigenfunctions of an operator L, and define $\Theta = (\theta_1, \ldots, \theta_n)$ and $\widehat{\Theta} = \left(\theta_j^{(i-1)}\right)$ for $i, j = 1, \ldots, n$, the $n \times n$ Wronskian matrix of $\theta_1, \ldots, \theta_n$, where ${}^{(k)}$ denotes the kth y-derivative.

To iterate the Darboux transformation, let $\theta_{[1]} = \theta_1$ and $\phi_{[1]} = \phi$ be a general eigenfunction of $L_{[1]} = L$, with $L_{[1]}$ covariant under the action of the Darboux transformation $G_{\theta_{[1]}} = \partial_y - \theta_{[1]}^{(1)}\theta_{[1]}^{-1}$. Then the general eigenfunctions $\phi_{[2]}$ for $L_{[2]} = G_{\theta_{[1]}} L_{[1]} G_{\theta_{[1]}}^{-1}$ are given by

$$\phi_{[2]} = G_{\theta_{[1]}}(\phi_{[1]}) = \phi_{[1]}^{(1)} - \theta_{[1]}^{(1)}\theta_{[1]}^{-1}\phi_{[1]}, \tag{4.1}$$

with

$$\theta_{[2]} = \phi_{[2]}|_{\phi \to \theta_2}. \tag{4.2}$$

Continuing this process, after n iterations ($n \geqslant 1$), the nth Darboux transformation of ϕ is given by

$$\phi_{[n+1]} = \phi_{[n]}^{(1)} - \theta_{[n]}^{(1)} \theta_{[n]}^{-1} \phi_{[n]}, \tag{4.3}$$

where

$$\theta_{[k]} = \phi_{[k]}|_{\phi \to \theta_k}. \tag{4.4}$$

4.2. Quasi-Wronskian solution of ncDS using Darboux transformations

We now determine the effect of the Darboux transformation $G_\theta = \partial_y - \theta_y \theta^{-1}$ on the Lax operator L given by (2.1*a*), with θ an eigenfunction of L. Corresponding results hold for the operator M given by (2.1*b*). L is covariant with respect to the Darboux transformation, and, by supposing that L is transformed to a new operator $\tilde{L}$, say, we calculate that the effect of the Darboux transformation $\tilde{L} = G_\theta L G_\theta^{-1}$ is such that

$$\tilde{\Lambda} = \Lambda - \sigma[J, \theta_y \theta^{-1}]. \tag{4.5}$$

Recalling that $\Lambda = [J, \sigma S]$, we have $\tilde{S} = S - \theta_y \theta^{-1}$, and hence, after n repeated Darboux transformations,

$$S_{[n+1]} = S_{[n]} - (\theta_{[n]})_y \theta_{[n]}^{-1}, \tag{4.6}$$

where $S_{[1]} = S, \theta_{[1]} = \theta$. We express $S_{[n+1]}$ in a quasideterminant form as

$$S_{[n+1]} = S + \begin{vmatrix} \theta_1 & \dots & \theta_n & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-2)} & \dots & \theta_n^{(n-2)} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \theta_1^{(n-1)} & \dots & \theta_n^{(n-1)} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \theta_1^{(n)} & \dots & \theta_n^{(n)} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}, \tag{4.7}$$

where $^{(k)}$ denotes the kth y-derivative. It should be noted here that each θ_i ($i = 1, \dots, n$) is not a single entry but a 2×2 matrix (since the θ_i are eigenfunctions of L, M). The Wronskian-like quasideterminant in (4.7) is termed a *quasi-Wronskian*, see [11], and [20] for details of Wronskian determinants.

For ease of notation, for integers $i, j = 1, \dots, n$, we denote by $Q(i, j)$ the quasideterminant [11]

$$Q(i, j) = \begin{vmatrix} \widehat{\Theta} & f_j & e_j \\ \Theta^{(n+i)} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}, \tag{4.8}$$

where, as before, $\widehat{\Theta} = \left(\theta_j^{(i-1)}\right)_{i,j=1,\dots,n}$ is the $n \times n$ Wronskian matrix of $\theta_1, \dots, \theta_n$ and $^{(k)}$ denotes the kth y-derivative, Θ is the row vector $(\theta_1, \dots, \theta_n)$ of length n, and f_j and e_j are $2n \times 1$ column vectors with a 1 in the $(2n - 2j - 1)$th and $(2n - 2j)$th row respectively and zeros elsewhere. Again each θ_i is a 2×2 matrix. In this definition of $Q(i, j)$, we allow i, j to take any integer values subject to the convention that if either $2n - 2j$ or $2n - 2j - 1$ lies outside the range $1, 2, \ldots, 2n$, then $e_j = f_j = 0$ and so $Q(i, j) = 0$. Hence (4.7) can be written as

$$S = S_0 + Q(0, 0), \tag{4.9}$$

where S_0 is any given solution of the ncDS equations. Here we choose the vacuum solution $S_0 = 0$ for simplicity.

It will be useful to express the quasi-Wronskian solution (4.9) in terms of the variables q and r, the variables in which our ncDS equations (2.5*a*)–(2.5*d*) are expressed. We have

$$S = Q(0, 0), \tag{4.10}$$

which gives, by applying the quasideterminant expansion formula (3.1) and expressing each θ_i $(i = 1, \ldots, n)$ as an appropriate 2×2 matrix

$$\theta_i = \begin{pmatrix} \phi_{2i-1} & \phi_{2i} \\ \psi_{2i-1} & \psi_{2i} \end{pmatrix} \tag{4.11}$$

for $\phi = \phi(x, y, t)$, $\psi = \psi(x, y, t)$, an expression for S in terms of quasi-Wronskians, namely

$$S = \begin{pmatrix} \begin{vmatrix} \widehat{\Theta} & f_0 \\ \phi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_0 \\ \phi^{(n)} & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \widehat{\Theta} & f_0 \\ \psi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_0 \\ \psi^{(n)} & \boxed{0} \end{vmatrix} \end{pmatrix}, \tag{4.12}$$

where $\phi^{(n)}$, $\psi^{(n)}$ denote the row vectors $\left(\phi_1^{(n)}, \ldots, \phi_{2n}^{(n)}\right)$, $\left(\psi_1^{(n)}, \ldots, \psi_{2n}^{(n)}\right)$, respectively. By comparing with (2.7), we immediately see that q, r can be expressed as quasi-Wronskians, namely

$$q = 2\sigma \begin{vmatrix} \widehat{\Theta} & e_0 \\ \phi^{(n)} & \boxed{0} \end{vmatrix}, \qquad r = -2\sigma \begin{vmatrix} \widehat{\Theta} & f_0 \\ \psi^{(n)} & \boxed{0} \end{vmatrix}. \tag{4.13}$$

4.3. Binary Darboux transformations

In order to define a binary Darboux transformation, we consider the adjoint Lax pair of the ncDS system (2.5*a*)–(2.5*d*). The notion of adjoint can be easily extended from the well-known matrix situation to any ring $\mathcal{R}$. An element $a \in \mathcal{R}$ has adjoint $a^\dagger$, where the adjoint has the following properties: if ∂ is a derivative acting on $\mathcal{R}$, $\partial^\dagger = -\partial$, and for any product AB of elements of, or operators on $\mathcal{R}$, $(AB)^\dagger = B^\dagger A^\dagger$. Thus the adjoint Lax pair for the ncDS system is given by

$$L^\dagger = -\partial_x - \Lambda^\dagger - \frac{1}{\sigma} J \partial_y, \tag{4.14a}$$

$$M^\dagger = -\partial_t - A^\dagger + \mathrm{i}\sigma\left(\Lambda_y^\dagger + \Lambda^\dagger \partial_y\right) + \mathrm{i} J \partial_{yy}. \tag{4.14b}$$

We construct a binary Darboux transformation in the usual manner (see, for example, [19]) by introducing a potential $\Omega(\phi, \psi)$ satisfying the relations

$$\Omega(\phi, \psi)_y = \psi^\dagger \phi, \tag{4.15a}$$

$$\Omega(\phi, \psi)_x = -\sigma \psi^\dagger J \phi, \tag{4.15b}$$

$$\Omega(\phi, \psi)_t = \mathrm{i}\left(\psi^\dagger J \phi_y - \psi_y^\dagger J \phi\right), \tag{4.15c}$$

with ϕ an eigenfunction of L, M and ψ an eigenfunction of $L^\dagger, M^\dagger$. This definition is also valid for non-scalar eigenfunctions; if Φ is an n-vector and Ψ an m-vector, then $\Omega(\Phi,\Psi)$ is an $m\times n$ matrix. We then define a binary Darboux transformation $G_{\theta,\rho}$ by

$$G_{\theta,\rho} = 1 - \theta\Omega(\theta,\rho)^{-1}\partial_y^{-1}\rho^\dagger \tag{4.16}$$

for eigenfunctions θ of L, M and ρ of $L^\dagger, M^\dagger$, so that

$$\phi_{[2]} = G_{\theta_{[1]},\rho_{[1]}}(\phi_{[1]}) = \phi_{[1]} - \theta_{[1]}\Omega(\theta_{[1]},\rho_{[1]})^{-1}\Omega(\rho_{[1]},\phi_{[1]}) \tag{4.17}$$

and

$$\psi_{[2]} = G_{\theta_{[1]},\rho_{[1]}}(\psi_{[1]}) = \psi_{[1]} - \rho_{[1]}\Omega(\rho_{[1]},\theta_{[1]})^{-\dagger}\Omega(\theta_{[1]},\psi_{[1]})^\dagger, \tag{4.18}$$

with

$$\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}, \qquad \rho_{[2]} = \psi_{[2]}|_{\psi\to\rho_2}. \tag{4.19}$$

After $n \geqslant 1$ iterations, the nth binary Darboux transformation is given by

$$\phi_{[n+1]} = \phi_{[n]} - \theta_{[n]}\Omega(\theta_{[n]},\rho_{[n]})^{-1}\Omega(\rho_{[n]},\phi_{[n]}) \tag{4.20}$$

and

$$\psi_{[n+1]} = \psi_{[n]} - \rho_{[n]}\Omega(\theta_{[n]},\rho_{[n]})^{-\dagger}\Omega(\theta_{[n]},\psi_{[n]})^\dagger, \tag{4.21}$$

with

$$\theta_{[n]} = \phi_{[n]}|_{\phi\to\theta_n}, \qquad \rho_{[n]} = \psi_{[n]}|_{\psi\to\rho_n}. \tag{4.22}$$

4.4. *Quasi-Grammian solution of ncDS using binary Darboux transformations*

We now determine the effect of the binary Darboux transformation $G_{\theta,\rho}$ on the operator L given by (2.1*a*), with θ an eigenfunction of L and ρ an eigenfunction of $L^\dagger$. Corresponding results hold for the operator M given by (2.1*b*) and its corresponding adjoint $M^\dagger$. The operator L is transformed to a new operator $\hat{L}$, say, where

$$\hat{L} = G_{\theta,\rho} L G_{\theta,\rho}^{-1}. \tag{4.23}$$

We find that $\hat{\Lambda} = \Lambda - \sigma[J, \theta_y\theta^{-1}] - \sigma[\hat{\theta}_y\hat{\theta}^{-1}, J]$, and hence, since $\Lambda = [J, \sigma S]$, it follows that

$$\hat{S} = S - \theta\Omega(\theta,\rho)^{-1}\rho^\dagger, \tag{4.24}$$

with $\hat{\theta} = -\theta\Omega(\theta,\rho)^{-1}$. After n repeated applications of the binary Darboux transformation $G_{\theta,\rho}$, we obtain

$$S_{[n+1]} = S_{[n]} - \sum_{k=1}^{n} \theta_{[k]}\Omega(\theta_{[k]},\rho_{[k]})^{-1}\rho_{[k]}^\dagger, \tag{4.25}$$

where $S_{[1]} = S$, $\theta_{[1]} = \theta$. Defining $\Theta = (\theta_1,\ldots,\theta_n)$ and $P = (\rho_1,\ldots,\rho_n)$, we express $S_{[n+1]}$ in the *quasi-Grammian* form [11] as

$$S_{[n+1]} = S + \begin{vmatrix} \Omega(\Theta,P) & P^\dagger \\ \Theta & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}, \tag{4.26}$$

where Ω is the Grammian-like matrix defined by (4.15*a*)–(4.15*c*). Note that, for $i = 1,\ldots,n$, each θ_i, ρ_i is a 2×2 matrix (since the θ_i, ρ_i are eigenfunctions of L, M and $L^\dagger, M^\dagger$, respectively).

For integers $i, j = 1, \ldots, n$, denote by $R(i, j)$ the quasi-Grammian [11]

$$R(i,j) = (-1)^j \begin{vmatrix} \Omega(\Theta, P) & P^{\dagger(j)} \\ \Theta^{(i)} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}, \tag{4.27}$$

so that, by once again choosing a trivial vacuum for simplicity, (4.26) can be expressed as

$$S = R(0,0). \tag{4.28}$$

As in the quasi-Wronskian case, we apply the quasideterminant expansion formula (3.1), choosing the matrices θ_i $(i = 1, \ldots, n)$ as in (4.11) and $P = \Theta H^\dagger$, where H is a constant square matrix, in this case $2n \times 2n$, which we assume to be invertible, with $H^\dagger$ denoting the Hermitian conjugate of H. Thus

$$S = \begin{pmatrix} \begin{vmatrix} \Omega(\Theta, P) & H\phi^\dagger \\ \phi & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & H\psi^\dagger \\ \phi & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \Omega(\Theta, P) & H\phi^\dagger \\ \psi & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & H\psi^\dagger \\ \psi & \boxed{0} \end{vmatrix} \end{pmatrix}, \tag{4.29}$$

where ϕ, ψ denote the row vectors $(\phi_1, \ldots, \phi_{2n})$, $(\psi_1, \ldots, \psi_{2n})$, respectively, which gives, by comparing the above matrix with (2.7), quasi-Grammian expressions for q, r, namely

$$q = 2\sigma \begin{vmatrix} \Omega(\Theta, P) & H\psi^\dagger \\ \phi & \boxed{0} \end{vmatrix}, \qquad r = -2\sigma \begin{vmatrix} \Omega(\Theta, P) & H\phi^\dagger \\ \psi & \boxed{0} \end{vmatrix}. \tag{4.30}$$

Thus we have obtained, in (4.13), expressions for q, r in terms of quasi-Wronskians, and in (4.30), expressions in terms of quasi-Grammians. We now show how these solutions can be verified by direct substitution by first explaining the procedure used to determine the derivative of a quasideterminant.

5. Derivatives of a quasideterminant

We consider a general quasideterminant of the form

$$\Xi = \begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix}, \tag{5.1}$$

where A, B, C and D are matrices of sizes $2n \times 2n$, $2n \times 2$, $2 \times 2n$ and 2×2, respectively. If A is a Grammian-like matrix with derivative

$$A' = \sum_{i=1}^{k} E_i F_i, \tag{5.2}$$

where E_i (F_i) are column (row) vectors of comparable lengths, it can be shown that (see appendix)

$$\Xi' = \begin{vmatrix} A & B \\ C' & \boxed{D'} \end{vmatrix} + \begin{vmatrix} A & B' \\ C & \boxed{O_2} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & E_i \\ C & \boxed{O_2} \end{vmatrix} \begin{vmatrix} A & B \\ F_i & \boxed{O_2} \end{vmatrix}, \tag{5.3}$$

where 'O_2' denotes the 2×2 zero matrix. If A does not have a Grammian-like structure, we find that

$$\Xi' = \begin{vmatrix} A & B \\ C' & \boxed{D'} \end{vmatrix} + \sum_{k=0}^{n-1} \begin{vmatrix} A & f_k & e_k \\ C & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix} \cdot \begin{vmatrix} A & B \\ \begin{matrix} (A^{2n-2k-1})' \\ (A^{2n-2k})' \end{matrix} & \boxed{\begin{matrix} (B^{2n-2k-1})' \\ (B^{2n-2k})' \end{matrix}} \end{vmatrix}, \tag{5.4}$$

where A^k denotes the kth row of A. Formulae (5.3) and (5.4) can be utilized to obtain expressions for the derivatives of the quasideterminants $Q(i, j)$ (detailed in the appendix), namely

$$Q(i, j)_y = Q(i+1, j) - Q(i, j+1) + Q(i, 0)Q(0, j), \tag{5.5a}$$

$$Q(i, j)_x = -\sigma\,(JQ(i+1, j) - Q(i, j+1)J + Q(i, 0)JQ(0, j)), \tag{5.5b}$$

$$Q(i, j)_t = \mathrm{i}\,(JQ(i+2, j) - Q(i, j+2)J + Q(i, 1)JQ(0, j) + Q(i, 0)JQ(1, j)). \tag{5.5c}$$

It turns out that the derivatives of $R(i, j)$ match exactly with those of $Q(i, j)$, hence subsequent calculations to verify the quasi-Wronskian solution of the ncDS equations will also be valid in the quasi-Grammian case, meaning that we need only verify one case.

6. Direct verification of quasi-Wronskian and quasi-Grammian solutions

We now show that

$$S = Q(0, 0) \qquad \text{and} \qquad S = \mathrm{R}(0, 0) \tag{6.1}$$

are solutions of the ncDS system (2.5a)–(2.5d), where S is the 2×2 matrix given by (2.7) and $\Lambda = [J, \sigma S]$. Using the derivatives of $Q(i, j)$ obtained in section 5, we have, on setting $i = j = 0$,

$$S_y = Q(1, 0) - Q(0, 1) + Q(0, 0)^2, \tag{6.2a}$$

$$S_x = -\sigma\,(JQ(1, 0) - Q(0, 1)J + Q(0, 0)JQ(0, 0)), \tag{6.2b}$$

$$S_t = \mathrm{i}\,(JQ(2, 0) - Q(0, 2)J + Q(0, 1)JQ(0, 0) + Q(0, 0)JQ(1, 0)), \tag{6.2c}$$

$$\begin{aligned} S_{yy} &= Q(2, 0) + Q(0, 2) - 2Q(1, 1) - Q(0, 1)Q(0, 0) + Q(0, 0)Q(1, 0) \\ &\quad + 2(Q(1, 0)Q(0, 0) - Q(0, 0)Q(0, 1)) + 2Q(0, 0)^3, \end{aligned} \tag{6.2d}$$

$$\begin{aligned} S_{xx} &= \sigma^2(Q(2, 0) + Q(0, 2) - 2JQ(1, 1)J - Q(0, 1)Q(0, 0) + Q(0, 0)Q(1, 0) \\ &\quad - 2(Q(0, 0)JQ(0, 1)J - JQ(1, 0)JQ(0, 0)) + 2Q(0, 0)JQ(0, 0)JQ(0, 0)). \end{aligned} \tag{6.2e}$$

Substituting the above in (2.9), all terms cancel exactly and thus the quasi-Wronskian solution $S = Q(0, 0)$ is verified. As mentioned previously, we obtain the same derivative formulae whether we use the quasi-Wronskian or quasi-Grammian formulation, and hence the above calculation also confirms the validity of the quasi-Grammian solution $S = R(0, 0)$.

7. Dromion solutions

To obtain dromion solutions of the system of ncDS equations, we use the quasi-Grammian solution $S = R(0, 0)$ rather than the quasi-Wronskian solution since verification of reality conditions is simpler.

7.1. (n, n)-dromion solution—noncommutative case

We modify the approach of [5], where dromion solutions of a system of commutative DS equations were determined. We consider the ncDS system (2.5a)–(2.5d) and, by specifying that certain parameters in the quasi-Grammian are of matrix rather than scalar form, we are able to obtain dromion solutions valid in the noncommutative case. Due to the complexity of this solution compared to the scalar case considered in [5], we look in some detail only at the simplest cases of the (1, 1)- and (2, 2)-dromion solutions. We do however verify reality for the general case. Note here that to obtain dromion solutions, we consider the DSI case, and hence choose $\sigma = -1$.

Recall the expressions for q, r obtained in terms of quasi-Grammians in (4.30), namely

$$q = -2\begin{vmatrix} \Omega(\Theta, P) & H\psi^{\dagger} \\ \phi & \boxed{0} \end{vmatrix}, \qquad r = 2\begin{vmatrix} \Omega(\Theta, P) & H\phi^{\dagger} \\ \psi & \boxed{0} \end{vmatrix}, \tag{7.1}$$

where ϕ, ψ denote the row vectors $(\phi_1, \dots, \phi_{2n}), (\psi_1, \dots, \psi_{2n})$ respectively and $H = (h_{ij})$ is a constant square invertible matrix, with † denoting conjugate transpose. By once again considering the dispersion relations for the system, we are able to choose expressions for ϕ, ψ corresponding to dromion solutions. From (A.10), (A.11) and the definition of $\Theta = (\theta_1, \dots, \theta_n)$, where θ_i is given by (4.11), it follows that ϕ, ψ satisfy the relations

$$(\phi_j)_x = (\phi_j)_y, \qquad (\phi_j)_t = \mathrm{i}(\phi_j)_{yy}, \tag{7.2a}$$

$$(\psi_j)_x = -(\psi_j)_y, \qquad (\psi_j)_t = -\mathrm{i}(\psi_j)_{yy}. \tag{7.2b}$$

(Since the dispersion relations for P are the same as those for Θ when $\sigma = -1$ (see (A.10), (A.11), (A.17), (A.18)), considering P rather than Θ and recalling that $P = \Theta H^{\dagger}$ will give the same relations (7.1)).

So far we have not specified the nature of the noncommutativity we are considering. One of the most straightforward cases to consider is to express our fields q and r as 2×2 matrices. Thus for dromion solutions in the noncommutative case, we choose

$$\phi_j = \alpha_j I_2, \tag{7.3a}$$

$$\psi_j = \beta_j I_2, \tag{7.3b}$$

where I_2 denotes the 2×2 identity matrix, and α_j, β_j the exponentials [4, 5]

$$\alpha_j = \exp\left(p_j x + \mathrm{i}p_j^2 t + p_j y + \alpha_{j_0}\right), \tag{7.4a}$$

$$\beta_j = \exp\left(q_j x - \mathrm{i}q_j^2 t - q_j y + \beta_{j_0}\right), \tag{7.4b}$$

for $j = 1, \dots, 2n$, suitable phase constants $\alpha_{j_0}, \beta_{j_0}$ and constants p_j, q_j, whose real parts are taken to be positive in order to give the correct asymptotic behaviour. The matrix H can be assumed to have unit diagonal since we are free to choose the phase constants $\alpha_{j_0}, \beta_{j_0}$ arbitrarily [5]. Using the coordinate transformation $X = x + y$, $Y = -(x - y)$, we have

$$\alpha_j = \exp\left(p_j X + \mathrm{i}p_j^2 t + \alpha_{j_0}\right), \tag{7.5a}$$

$$\beta_j = \exp\left(-q_j Y - \mathrm{i}q_j^2 t + \beta_{j_0}\right), \tag{7.5b}$$

so that $\alpha_j = \alpha_j(X, t)$, $\beta_j = \beta_j(Y, t)$.

We now choose to simplify our notation so that we are working with only $\phi_1, \dots, \phi_n$ and $\psi_1, \dots, \psi_n$ by relabelling ϕ_j as $\phi_{\frac{j+1}{2}}$ for odd j (i.e. $j = 1, 3, \dots, 2n - 1$) and setting $\phi_j = 0$ for even j ($j = 0, 2, \dots, 2n$), and similarly relabelling ψ_j as $\psi_{\frac{j}{2}}$ for even j and setting $\psi_j = 0$ for odd j, so that $\theta_j = \mathrm{diag}(\phi_j, \psi_j)$ $(j = 1, \dots, n)$ and

$$\phi = (\phi_1 \quad 0 \quad \phi_2 \quad 0 \quad \ldots \quad \phi_n \quad 0), \tag{7.6a}$$

$$\psi = (0 \quad \psi_1 \quad 0 \quad \psi_2 \quad \ldots \quad 0 \quad \psi_n), \tag{7.6b}$$

where each ϕ_j, ψ_j is a 2×2 matrix as defined in (7.1) above. Thus, for $n = 1$, q (which we henceforth denote by q^1 for the $(1, 1)$-dromion case and q^n for the (n, n)-dromion case) can be expressed in the quasi-Grammian form as

$$q^1 = -2 \begin{vmatrix} \Omega(\Theta, P) & H \begin{matrix} 0 & 0 \\ 0 & 0 \\ \beta_1^* & 0 \\ 0 & \beta_1^* \end{matrix} \\ \begin{matrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}, \tag{7.7}$$

where $H = (h_{ij})$ is a constant invertible 4×4 matrix. Applying the quasideterminant expansion formula (3.1) allows us to express q^1 as a 2×2 matrix, where each entry is a quasi-Grammian, namely

$$q^1 = -2 \begin{pmatrix} \begin{vmatrix} \Omega(\Theta, P) & \begin{matrix} h_{13}\beta_1^* \\ \vdots \\ h_{43}\beta_1^* \end{matrix} \\ \alpha_1 \quad 0 \quad 0 \quad 0 & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \begin{matrix} h_{14}\beta_1^* \\ \vdots \\ h_{44}\beta_1^* \end{matrix} \\ \alpha_1 \quad 0 \quad 0 \quad 0 & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \Omega(\Theta, P) & \begin{matrix} h_{13}\beta_1^* \\ \vdots \\ h_{43}\beta_1^* \end{matrix} \\ 0 \quad \alpha_1 \quad 0 \quad 0 & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \begin{matrix} h_{14}\beta_1^* \\ \vdots \\ h_{44}\beta_1^* \end{matrix} \\ 0 \quad \alpha_1 \quad 0 \quad 0 & \boxed{0} \end{vmatrix} \end{pmatrix}$$

$$= -2 \begin{pmatrix} q_{11}^1 & q_{12}^1 \\ q_{21}^1 & q_{22}^1 \end{pmatrix}, \text{ say.} \tag{7.8}$$

We consider each quasi-Grammian in turn, but as an example we will look at the quasi-Grammian q_{11}^1. We apply (3.5) to express q_{11}^1 as a ratio of determinants, namely

$$q_{11}^1 = -2 \frac{\begin{vmatrix} \Omega(\Theta, P) & \begin{matrix} h_{13}\beta_1^* \\ \vdots \\ h_{43}\beta_1^* \end{matrix} \\ \alpha_1 \quad 0 \quad 0 \quad 0 & 0 \end{vmatrix}}{|\Omega(\Theta, P)|} = -2 \frac{G_{11}^1}{F}, \text{ say.} \tag{7.9}$$

(We have introduced the notation G_{vw}^n and q_{vw}^n $(v, w = 1, 2)$ to emphasize that we are considering the (v, w)th entry of the expansion of q^n in the (n, n)-dromion case). Although (3.5) is valid only in the commutative case, our assumption here is that the variables q, r in our system of DS equations, and also the parameters ϕ_j, ψ_j, are noncommutative. The exponentials α_j, β_j given by (7.1) are clearly commutative by definition, hence we are free to use the result (3.5).

By expanding the quasi-Grammian r in (7.1) in a similar way and extracting the (1, 1)th entry, we obtain

$$r_{11}^{1} = 2\frac{\begin{vmatrix} & & & & h_{11}\alpha_1^* \\ & \Omega(\Theta, P) & & & \vdots \\ & & & & h_{41}\alpha_1^* \\ 0 & 0 & \beta_1 & 0 & 0 \end{vmatrix}}{|\Omega(\Theta, P)|} = 2\frac{K_{11}^{1}}{F}, \text{ say,} \tag{7.10}$$

with similar results for q_{12}^1, r_{12}^1, etc.

7.2. Reality conditions

In the (n, n)-dromion case, we have

$$q_{vw}^{n} = -2\frac{G_{vw}^{n}}{F}, \tag{7.11a}$$

$$r_{vw}^{n} = 2\frac{K_{vw}^{n}}{F}, \tag{7.11b}$$

for $v, w = 1, 2$. To verify reality, we must check that $r_{vw}^n = \pm(q_{vw}^n)^*$, i.e. that F is real, and $(G_{vw}^n)^* = \pm K_{vw}^n$, where $(G_{vw}^n)^*$ denotes the complex conjugate of G_{vw}^n.

We do not give details here, however we can show that F can be expressed in the form

$$F = |I_{4n} + H\Phi|, \tag{7.12}$$

where H is $4n \times 4n$, constant and invertible, and Φ is the $4n \times 4n$ matrix

$$\Phi = \begin{pmatrix} \int_{-\infty}^{X} \phi_1^*\phi_1 \, \mathrm{d}X & O_2 & \dots & \int_{-\infty}^{X} \phi_1^*\phi_n \, \mathrm{d}X & O_2 \\ O_2 & \int_{Y}^{\infty} \psi_1^*\psi_1 \, \mathrm{d}Y & \dots & O_2 & \int_{Y}^{\infty} \psi_1^*\psi_n \, \mathrm{d}Y \\ \vdots & \vdots & & \vdots & \vdots \\ \int_{-\infty}^{X} \phi_n^*\phi_1 \, \mathrm{d}X & O_2 & \dots & \int_{-\infty}^{X} \phi_n^*\phi_n \, \mathrm{d}X & O_2 \\ O_2 & \int_{Y}^{\infty} \psi_n^*\psi_1 \, \mathrm{d}Y & \dots & O_2 & \int_{Y}^{\infty} \psi_n^*\psi_n \, \mathrm{d}Y \end{pmatrix}, \tag{7.13}$$

remembering that each ϕ_i, ψ_i $(i = 1, \dots, n)$ is a 2×2 matrix and O_2 denotes the 2×2 zero matrix. With F in this form, it is straightforward to show that F is real so long as H is a Hermitian matrix (see [21]). This condition is also required for $(G_{vw}^n)^* = \pm K_{vw}^n$—in fact, we find that $r_{vw}^n = -(q_{vw}^n)^*$ so long as H is Hermitian. This agrees with the work of Gilson and Nimmo in [5].

7.3. (1, 1)-*dromion solution—matrix case*

We now show computer plots of the (1, 1)-dromion solution in the noncommutative case, where we choose ϕ_1, ψ_1 to be 2×2 matrices as in (7.1). A suitable choice of the parameters p_1, q_1 and of the 4×4 Hermitian matrix H allows us to obtain plots of the four quasi-Grammian solutions $q_{11}^1, q_{12}^1, q_{21}^1, q_{22}^1$ as detailed in (7.8).

We are restricted in our choice of p_1, q_1 and H in that we require $F \neq 0$. In particular, we derive conditions so that $F > 0$. The determinant F can be expanded in terms of minor matrices of $H = (h_{ij})$ $(i, j = 1, \dots, 4)$, giving

$$\begin{aligned} F = 1 &+ P_1\left(h_{234}^{234} + h_{134}^{134}\right)\mathrm{e}^{2\eta} + Q_1\left(h_{124}^{124} + h_{123}^{123}\right)\mathrm{e}^{-2\xi} + P_1^2 h_{34}^{34}\,\mathrm{e}^{4\eta} \\ &+ Q_1^2 h_{12}^{12}\,\mathrm{e}^{-4\xi} + P_1 Q_1\left(h_{24}^{24} + h_{14}^{14} + h_{23}^{23} + h_{13}^{13}\right)\mathrm{e}^{2\eta-2\xi} \\ &+ P_1^2 Q_1\left(h_4^4 + h_3^3\right)\mathrm{e}^{4\eta-2\xi} + P_1 Q_1^2\left(h_2^2 + h_1^1\right)\mathrm{e}^{2\eta-4\xi} + P_1^2 Q_1^2 h\,\mathrm{e}^{4\eta-4\xi}, \end{aligned} \tag{7.14}$$

where we define $P_1 = 1/(2\,\mathrm{Re}(p_1))$, $Q_1 = 1/(2\,\mathrm{Re}(q_1))$, $\eta = \mathrm{Re}\,(p_1)(X - 2\,\mathrm{Im}(p_1)t)$ and $\xi = \mathrm{Re}\,(q_1)(Y - 2\,\mathrm{Im}(q_1)t)$, with $h^{rs\ldots}_{ij\ldots}$ denoting the minor matrix obtained by removing rows $i, j, \ldots$ and columns $r, s, \ldots$ of H, where $i, j, \ldots, r, s, \ldots \in \{1, 2, 3, 4\}$. We have used '$h$' to indicate that no rows or columns have been removed, i.e. $h = \det H$. We can also obtain expressions for each G^1_{vw} $(v, w = 1, 2)$, for instance

$$G^1_{11} = \alpha_1\beta_1^*\left(h^{124}_{234} - P_1 h^{14}_{34}\,\mathrm{e}^{2\eta} - Q_1 h^{12}_{23}\,\mathrm{e}^{-2\xi} - P_1 Q_1 h^1_3\,\mathrm{e}^{2\eta-2\xi}\right), \tag{7.15}$$

with α_1, β_1 defined as in (7.1). Similar expansions can be obtained for G^1_{12}, G^1_{21} and G^1_{22}. Thus, it can be seen that for $F > 0$, we require $\mathrm{Re}(p_1)$, $\mathrm{Re}(q_1)$ and each of the minor matrices in the expansion of F to be greater than zero. Suitable choices of p_1, q_1 and H have been used to obtain the dromion plots shown later. In addition to the matrix-valued fields q and r, there are also matrix-valued fields A_1 and A_2 in the ncDS system (2.5*a*)–(2.5*d*). Plotting the derivatives of these fields gives plane waves as follows.

From (4.29), we have an expression for the 2×2 matrix S in terms of quasi-Grammians. By (2.8), $A = -\mathrm{i}S_x - \mathrm{i}JS_y$, therefore substituting for S using (4.29) and equating matrix entries gives quasi-Grammian expressions for A_1, A_2, namely

$$A_1 = -\mathrm{i}\begin{vmatrix} \Omega & H\phi^\dagger \\ \phi & \boxed{0} \end{vmatrix}_x - \mathrm{i}\begin{vmatrix} \Omega & H\phi^\dagger \\ \phi & \boxed{0} \end{vmatrix}_y, \tag{7.16a}$$

$$A_2 = -\mathrm{i}\begin{vmatrix} \Omega & H\psi^\dagger \\ \psi & \boxed{0} \end{vmatrix}_x + \mathrm{i}\begin{vmatrix} \Omega & H\psi^\dagger \\ \psi & \boxed{0} \end{vmatrix}_y. \tag{7.16b}$$

We choose ϕ_1, ψ_1 to be 2×2 matrices as before so that the boxed expansion element '0' in each quasi-Grammian is the 2×2 zero matrix. Thus, expanding each quasi-Grammian in the usual manner gives a 2×2 matrix where each entry is a quasi-Grammian, and hence we have distinct expressions for A_1, A_2 corresponding to each of the four dromions $q^1_{11}, q^1_{12}, q^1_{21}, q^1_{22}$. Considering (2.5*c*)–(2.5*d*), we find that plotting the combination $(\partial_x + \sigma\partial_y)A_1$ for $\sigma = -1$ gives a plane wave travelling in the X-direction, while the combination $(\partial_x - \sigma\partial_y)A_2$ gives a plane wave in the Y-direction. These, along with the dromions corresponding to each plane wave, have been plotted at time $t = 0$ in figures 1 and 2.

As can be seen from figure 1, single dromions of differing heights occur in each of the fields q_{11}, q_{12}, q_{21} and q_{22}. If we were to plot the (1, 1)-dromion solution in the commutative (scalar) case (that is, if we were to choose q and its complex conjugate to be of scalar rather than matrix form), we would obtain only one dromion in the single field q. This dromion and its plane waves would have the same basic structure as those above, and thus there would be no marked difference in the appearance of the dromions in the commutative and noncommutative cases. The main difference between the two situations concerns the number of parameters—a far greater number in the noncommutative case gives us more freedom to control the heights of the dromions, however some extra care has to be taken in choosing the parameters so that no singularities occur in the solution.

7.4. (2, 2)-*dromion solution—matrix case*

In the scalar case [5], Gilson and Nimmo carried out a detailed asymptotic analysis of their (M, N)-dromion solution, and were able to obtain compact expressions for the phase-shifts and changes in amplitude that occur due to dromion interactions. They then used the results of this analysis to study a class of (2, 2)-dromions with scattering-type interaction properties.

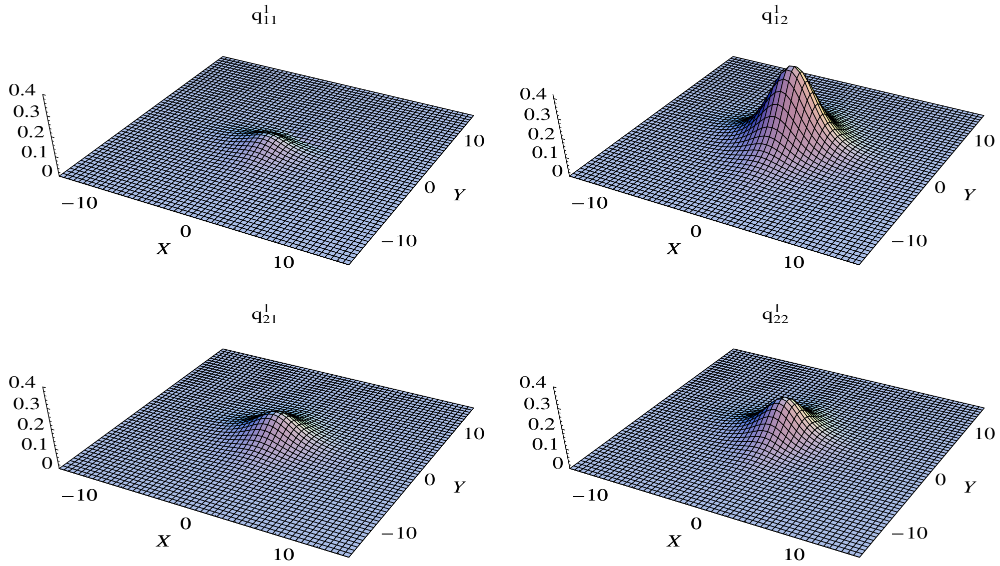

Figure 1. (1, 1)-dromion plots with $p_1 = \frac{1}{2} + \mathrm{i}$, $q_1 = \frac{1}{2} - \mathrm{i}$ and $h_{12} = \frac{1}{2}$, $h_{13} = \frac{1}{4}$, $h_{14} = \frac{3}{4}$, $h_{23} = \frac{1}{3}$, $h_{24} = \frac{1}{2}$, $h_{34} = \frac{1}{3}$.

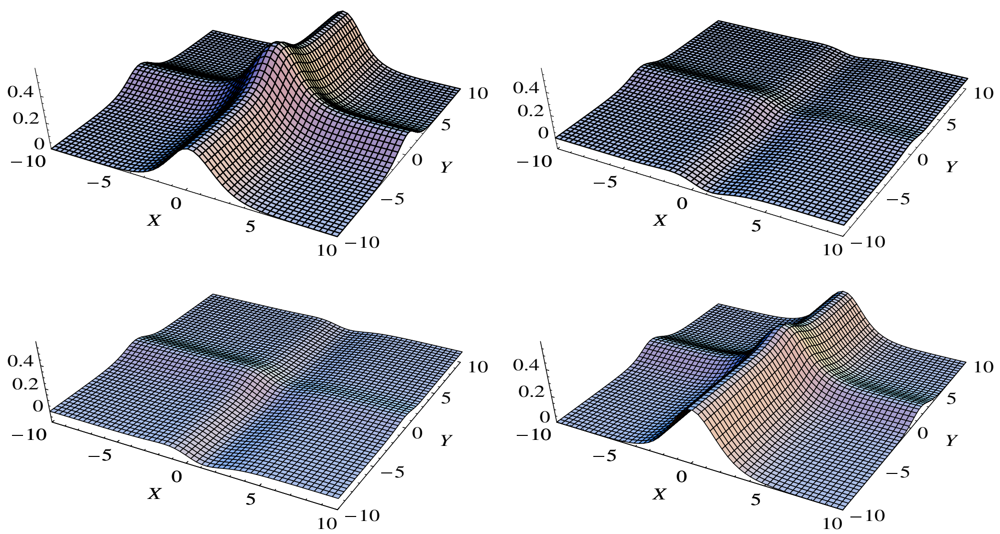

Figure 2. Plane waves corresponding to, clockwise from top left, q^1_{11}, q^1_{12}, q^1_{22}, q^1_{21}.

The Hermitian matrix H could be chosen in such a way so that some of the dromions had zero amplitude either as $t \to -\infty$ or as $t \to +\infty$.

For the (2, 2)-dromion solution in the matrix case, detailed calculations of this type are more complicated due to the large number of terms involved. However, we can adopt the same approach to carry out some of the more straightforward calculations. In particular, we obtain plots of the situation in which the (1, 1)th dromion in each of the solutions q^2_{11}, q^2_{12}, q^2_{21} and q^2_{22} does not appear as $t \to -\infty$. These are depicted in figures 3–5.

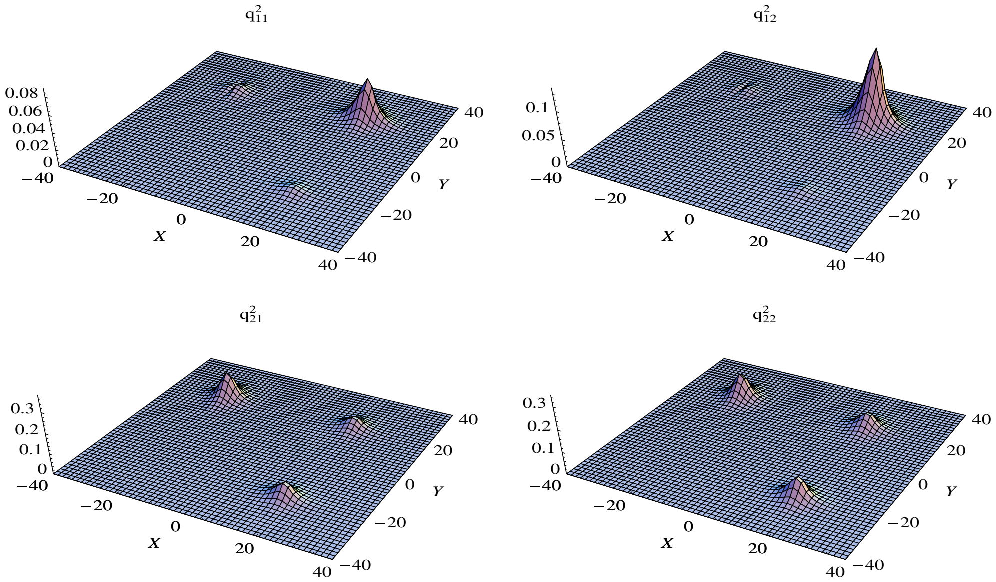

Figure 3. (2, 2)-dromion plots at $t = -10$.

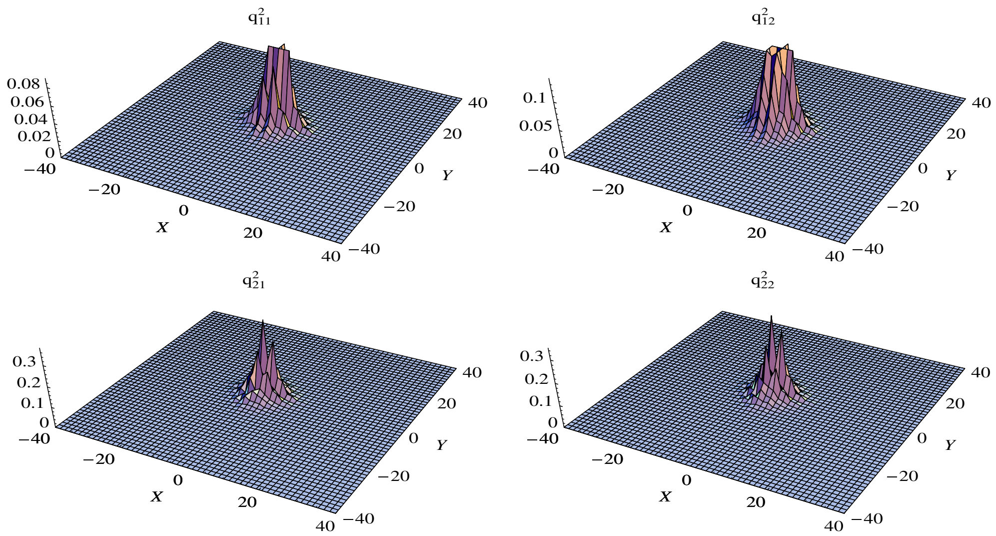

Figure 4. (2, 2)-dromion plots at $t = 0$.

To analyse this situation, we focus our attention on q_{11}^2 and consider G_{11}^2 in a frame moving with the (1, 1)th dromion. We define

$$\hat{X} = X - 2\,\mathrm{Im}(p_1)t, \tag{7.17a}$$

$$\hat{Y} = Y - 2\,\mathrm{Im}(q_1)t, \tag{7.17b}$$

and consider the limits of G_{11}^2 as $t \to -\infty$. Let

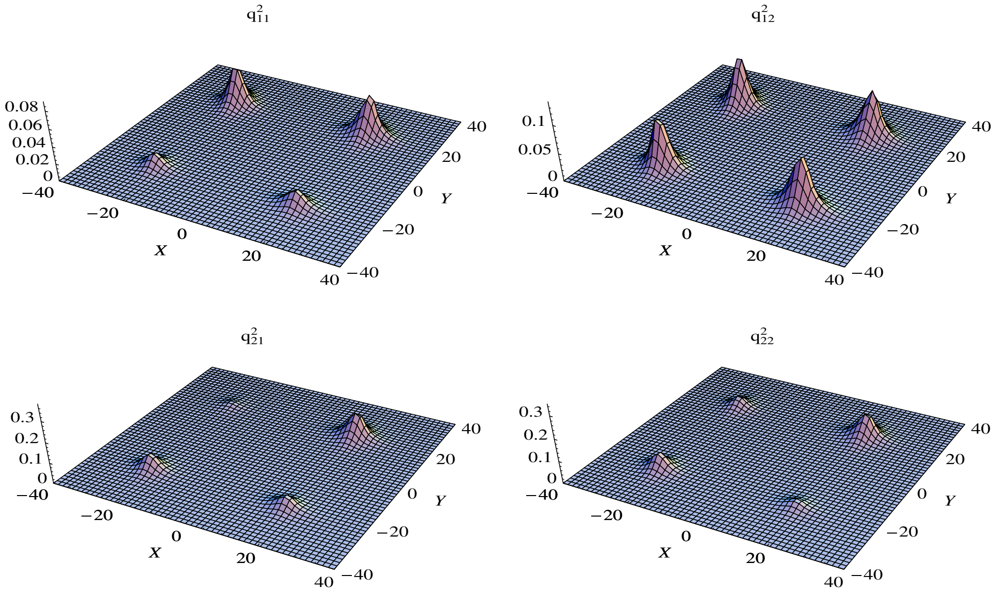

Figure 5. (2, 2)-dromion plots at $t = 10$.

$$\begin{aligned}\eta_2 &= \mathrm{Re}(p_2)\,(X - 2\,\mathrm{Im}(p_2)t)\\ &= \mathrm{Re}(p_2)(\hat{X} - 2(\mathrm{Im}(p_2) - \mathrm{Im}(p_1))t),\end{aligned} \tag{7.18a}$$

and similarly

$$\begin{aligned}\xi_2 &= \mathrm{Re}(q_2)(Y - 2\,\mathrm{Im}(q_2)t)\\ &= \mathrm{Re}(q_2)(\hat{Y} - 2(\mathrm{Im}(q_2) - \mathrm{Im}(q_1))t).\end{aligned} \tag{7.18b}$$

We choose to order the p_i, q_i $(i = 1, 2)$ by means of their imaginary parts, so that $\mathrm{Im}(p_1) > \mathrm{Im}(p_2)$ and $\mathrm{Im}(q_1) < \mathrm{Im}(q_2)$. Thus, as $t \to -\infty, \eta_2 \to -\infty$ and $\xi_2 \to +\infty$. It can easily be shown that $\eta_2, -\xi_2$ determine the real parts of the exponents in α_2, β_2 respectively, where α_2, β_2 are defined as in (7.1), so that, as $t \to -\infty, \alpha_2, \beta_2 \to 0$ (and hence $\alpha_2^*, \beta_2^* \to 0$ also). Therefore, by setting $\alpha_2, \alpha_2^*, \beta_2, \beta_2^* \to 0$ in G_{11}^2 and expanding the resulting determinant, we obtain a compact expression for G_{11}^2 as $t \to -\infty$, namely

$$G_{11}^2 = -\alpha_1\beta_1^*\left(h_{2345678}^{1245678} - P_1 h_{345678}^{145678}\,\mathrm{e}^{2\eta} - Q_1 h_{235678}^{125678}\,\mathrm{e}^{-2\xi} + P_1 Q_1 h_{35678}^{15678}\,\mathrm{e}^{2\eta-2\xi}\right). \tag{7.19}$$

Similar expressions can be obtained for the other three determinants G_{12}^2, G_{21}^2 and G_{22}^2 by considering an extension of (7.8) to the (2, 2)-dromion case and interchanging columns appropriately: for example, we interchange columns 3 and 4, and 7 and 8, in the expansion of G_{11}^2 as $t \to -\infty$ to obtain an analogous expression for G_{12}^2. Thus we have, as $t \to -\infty$, compact expressions for the minors of H governing the (1, 1)th dromion in each of $G_{11}^2, G_{12}^2, G_{21}^2, G_{22}^2$, namely

$$\begin{pmatrix} G_{11}^2 & G_{12}^2 \\ G_{21}^2 & G_{22}^2 \end{pmatrix} = -\alpha_1\beta_1^* \left\{ \begin{pmatrix} |h_{13}| & |h_{14}| \\ |h_{23}| & |h_{24}| \end{pmatrix} + P_1 T \begin{pmatrix} \begin{vmatrix} h_{12} & h_{13} \\ h_{22} & h_{23} \end{vmatrix} & \begin{vmatrix} h_{12} & h_{14} \\ h_{22} & h_{24} \end{vmatrix} \\ \begin{vmatrix} h_{11} & h_{13} \\ h_{21} & h_{23} \end{vmatrix} & \begin{vmatrix} h_{11} & h_{14} \\ h_{21} & h_{24} \end{vmatrix} \end{pmatrix} \mathrm{e}^{2\eta} \right.$$

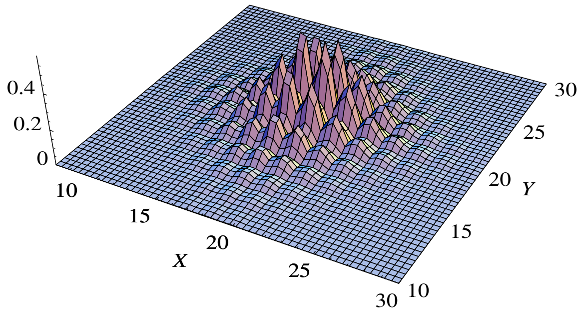

Figure 6. Detail of the q_{22}^2 dromion interaction shown in figure 4.

$$+\,Q_1T\begin{pmatrix}\begin{vmatrix}h_{13} & h_{14}\\ h_{43} & h_{44}\end{vmatrix} & \begin{vmatrix}h_{13} & h_{14}\\ h_{33} & h_{34}\end{vmatrix}\\ \begin{vmatrix}h_{23} & h_{24}\\ h_{43} & h_{44}\end{vmatrix} & \begin{vmatrix}h_{23} & h_{24}\\ h_{33} & h_{34}\end{vmatrix}\end{pmatrix}\mathrm{e}^{-2\xi}$$

$$\left.+P_1Q_1T\begin{pmatrix}\begin{vmatrix}h_{12} & h_{13} & h_{14}\\ h_{22} & h_{23} & h_{24}\\ h_{42} & h_{43} & h_{44}\end{vmatrix} & \begin{vmatrix}h_{12} & h_{13} & h_{14}\\ h_{22} & h_{23} & h_{24}\\ h_{32} & h_{33} & h_{34}\end{vmatrix}\\ \begin{vmatrix}h_{11} & h_{13} & h_{14}\\ h_{21} & h_{23} & h_{24}\\ h_{41} & h_{43} & h_{44}\end{vmatrix} & \begin{vmatrix}h_{11} & h_{13} & h_{14}\\ h_{21} & h_{23} & h_{24}\\ h_{31} & h_{33} & h_{34}\end{vmatrix}\end{pmatrix}T\mathrm{e}^{2\eta-2\xi}\right\}, \tag{7.20}$$

where $T = \mathrm{diag}(-1, 1)$. Since we have written out each minor matrix explicitly, rather than using the abbreviated notation as in (7.19) above, it can easily be seen that, by setting each of $h_{13}, h_{14}, h_{23}, h_{24}$ equal to zero, the $(1, 1)$th dromion in each of $q_{11}^2, q_{12}^2, q_{21}^2$ and q_{22}^2 will vanish as $t \to -\infty$. This is shown in figures 3–5, where we have chosen $p_1 = q_2 = \frac{1}{2} + \mathrm{i}$, $p_2 = q_1 = \frac{1}{2} - \mathrm{i}$, and appropriate values of h_{ij} $(i, j = 1, \ldots, 8)$. (Note that in the plots of q_{11}^2 and q_{12}^2 in figure 3, two of the dromions have very small amplitude). We have also shown, in figure 6, a close-up of one of the dromion interactions at $t = 0$.

8. Conclusions

In this paper we have derived a noncommutative version of the Davey–Stewartson equations and verified their quasi-Wronskian and quasi-Grammian solutions by direct substitution. The quasi-Grammian solution has then been used to obtain dromion solutions in the matrix case, which, if we were to consider these solutions in the scalar case, agree with the results of Gilson and Nimmo in [5]. We have obtained plots of the $(1, 1)$-dromion solution and its plane waves, choosing the entries of the Hermitian matrix H in such a way that our solution is always well defined. In the $(2, 2)$-dromion case, some of the more straightforward asymptotic calculations have been carried out, enabling us to obtain plots of the situation with one dromion vanishing as $t \to -\infty$.

Acknowledgments

Susan Macfarlane would like to thank the Engineering and Physical Sciences Research Council for a Research Studentship, and the reviewers of this paper for helpful comments.

Appendix

Here we prove the results of section 5. Consider a general quasideterminant of the form given by (5.1). Using the product rule for derivatives

$$\Xi' = D' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B'. \tag{A.1}$$

We modify slightly the approach of [11] and find that, if A is a Grammian-like matrix with derivative

$$A' = \sum_{i=1}^{k} E_i F_i, \tag{A.2}$$

where E_i (F_i) are column (row) vectors of comparable lengths, then

$$\begin{aligned}\Xi' &= D' - CA^{-1}B + \sum_{i=1}^{k}(CA^{-1}E_i)(F_iA^{-1}B) - CA^{-1}B' \\ &= \begin{vmatrix} A & B \\ C' & \boxed{D'} \end{vmatrix} + \begin{vmatrix} A & B' \\ C & \boxed{O_2} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & E_i \\ C & \boxed{O_2} \end{vmatrix} \begin{vmatrix} A & B \\ F_i & \boxed{O_2} \end{vmatrix},\end{aligned} \tag{A.3}$$

where O_2 denotes the 2×2 zero matrix. If, on the other hand, the matrix A does not have a Grammian-like structure, we can once again write the derivative Ξ' as a product of quasideterminants as above by inserting the $2n \times 2n$ identity matrix in the form

$$I = \sum_{k=0}^{n-1}(f_k e_k)(f_k e_k)^T, \tag{A.4}$$

with e_k, f_k defined as before, so that $(f_k e_k)$ denotes the $2n \times 2$ matrix with the $(2n-2k)$th and $(2n-2k-1)$th entries equal to 1 and every other entry zero. Then we find that

$$\Xi' = \begin{vmatrix} A & B \\ C' & \boxed{D'} \end{vmatrix} + \sum_{k=0}^{n-1} \begin{vmatrix} A & f_k \quad e_k \\ C & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix} \cdot \begin{vmatrix} A & B \\ \begin{matrix} (A^{2n-2k-1})' \\ (A^{2n-2k})' \end{matrix} & \boxed{\begin{matrix} (B^{2n-2k-1})' \\ (B^{2n-2k})' \end{matrix}} \end{vmatrix}, \tag{A.5}$$

where A^k denotes the kth row of A. It is also possible to obtain a column version of the derivative formula by inserting the identity in a different position. We now use formulae (A.3) and (A.5) to derive expressions for the derivatives of the quasideterminants $Q(i, j)$, $R(i, j)$.

Consider the quasi-Wronskian $Q(i, j)$ defined in section 4.2, namely

$$Q(i, j) = \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ \Theta^{(n+i)} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}. \tag{A.6}$$

Calculation of the derivatives of $Q(i, j)$ requires knowledge of the following result [11], that for arbitrarily large n

$$Q(i, j) = \begin{cases} -I_2 & i + j + 1 = 0 \\ O_2 & i < 0 \quad \text{or} \quad j < 0 \quad \text{and} \quad i + j + 1 \neq 0, \end{cases} \tag{A.7}$$

where O_2 denotes the 2×2 zero matrix. We utilize the dispersion relations for the ncDS system (2.5*a*)–(2.5*d*), found by considering the Lax pairs (2.1*a*)–(2.1*b*) in the trivial vacuum case, giving, for θ an eigenfunction of L, M,

$$\theta_x = -\sigma J \theta_y, \tag{A.8}$$

$$\theta_t = \mathrm{i}J\theta_{yy}, \tag{A.9}$$

and since $\Theta = (\theta_1, \ldots, \theta_n)$, it follows that

$$\Theta_x = -\sigma J\Theta_y, \tag{A.10}$$

$$\Theta_t = \mathrm{i}J\Theta_{yy}. \tag{A.11}$$

Thus, using (A.5), we have

$$Q(i, j)_y = Q(i+1, j) + \sum_{k=0}^{n-1} Q(i, k)Q(-k, j), \tag{A.12}$$

$$Q(i, j)_x = -\sigma\left(JQ(i+1, j) + \sum_{k=0}^{n-1} Q(i, k)JQ(-k, j)\right), \tag{A.13}$$

$$Q(i, j)_t = \mathrm{i}\left(JQ(i+2, j) + \sum_{k=0}^{n-1} Q(i, k)JQ(1-k, j)\right). \tag{A.14}$$

These can be simplified using (A.7), leaving the derivatives as given in (5.5*a*)–(5.5*c*).

We can apply a similar procedure to determine the derivatives of the quasi-Grammian $R(i, j)$ defined in section 4.4. The dispersion relations are found by considering the adjoint Lax pairs (4.14*a*), (4.14*b*) in the trivial vacuum case, giving, for ρ an eigenfunction of $L^\dagger$, $M^\dagger$

$$\rho_x = -\frac{1}{\sigma}J\rho_y, \tag{A.15}$$

$$\rho_t = \mathrm{i}J\rho_{yy}, \tag{A.16}$$

and since $P = (\rho_1, \ldots, \rho_n)$, we have

$$P_x = -\frac{1}{\sigma}JP_y, \tag{A.17}$$

$$P_t = \mathrm{i}JP_{yy}. \tag{A.18}$$

We also recall that from our construction of the binary Darboux transformation in section 4.3, the potential $\Omega(\phi, \psi)$ satisfies the relations (4.15*a*)–(4.15*c*), from which it follows that

$$\Omega(\Theta, P)_y = P^\dagger\Theta, \tag{A.19}$$

$$\Omega(\Theta, P)_x = -\sigma P^\dagger J\Theta, \tag{A.20}$$

$$\Omega(\Theta, P)_t = \mathrm{i}(P^\dagger J\Theta^{(1)} - P^{\dagger(1)}J\Theta), \tag{A.21}$$

where $^{(k)}$ denotes the kth y-derivative. Thus, using (A.3), we calculate the derivatives of $R(i, j)$ and find that they are identical to those of $Q(i, j)$ in (5.5*a*)–(5.5*c*).

References

[1] Davey A and Stewartson K 1974 *Proc. R. Soc. Lond.* A **338** 101–10
[2] Boiti M, Leon J J-P, Martina L and Pempinelli F 1988 *Phys. Lett.* A **132** 432–39
[3] Fokas A S and Santini P M 1989 *Phys. Rev. Lett.* **63** 1329–34

[4] Hietarinta J and Hirota R 1990 *Phys. Lett.* A **145** 237–44
[5] Gilson C R and Nimmo J J C 1991 *Proc. R. Soc. Lond.* A **435** 339–57
[6] Hamanaka M 2005 *Phys. Lett.* B **625** 324–32
[7] Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99–124
[8] Dimakis A and Müller-Hoissen F 2009 *SIGMA* **5** 1–18
[9] Lechtenfeld O and Popov A D 2001 *J. High Energy Phys.* JHEP11(2001)040
[10] Lechtenfeld O, Mazzanti L, Penati S, Popov A D and Tamassia L 2005 *Nucl. Phys. B* **705** 477–503
[11] Gilson C R and Nimmo J J C 2007 *J. Phys. A: Math. Theor.* **40** 3839–50
[12] Schultz C L, Ablowitz M J and Bar Yaacov D 1987 *Phys. Rev. Lett.* **59** 2825–28
[13] Ablowitz M J and Schultz C L 1989 *Phys. Lett.* A **135** 433–37
[14] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[15] Li C X 2008 private communication
[16] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
[17] Gelfand I, Gelfand S, Retakh V and Wilson R L 2005 *Adv. Math.* **193** 56–141
[18] Gelfand I and Retakh V 2005 *Funct. Anal. Appl.* **25** 91–102
[19] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons (Springer Series in Nonlinear Dynamics)* (Berlin: Springer)
[20] Freeman N C and Nimmo J J C 1983 *Phys. Lett.* **1** 1–3
[21] Ratter M C 1998 Grammians in nonlinear evolution equations *PhD Thesis*